Numerical Analysis for ODEs

Wintersemester 2023/24

Initial Value Problems

Let $\alpha, b \in \mathbb{R}$, $\alpha < b$, $f: [\alpha, b] \times \mathbb{R} \to \mathbb{R}$ a function and $y_0 \in \mathbb{R}$.
The model initial value problem (IVP) is the following:

(1.1) We seek function $y : [\alpha, b] \to \mathbb{R}$ such that

$$\begin{cases} y'(t) = f(t, y(t)) & , \alpha \leq t \leq b. \\ y(\alpha) = y_0. \end{cases} \tag{1.1}$$

Let that f is a continuous function for $(t, y) \in [\alpha, b] \times \mathbb{R}$, we write $f \in C([\alpha, b] \times \mathbb{R})$. Every function $y \in C^1[\alpha, b]$ that satisfies the differential equation of (1.1) and its initial condition $y(\alpha) = y_0$, is called as a solution of the IVP (1.1).

1.1 Existence and Uniqueness

First, we will study the case where f is a polynomial of first order with respect to y, then the ODE is called linear and the (1.1) may written as

$$\begin{cases} y'(t) = p(t) y(t) + q(t) & , \alpha \leq t \leq b. \\ y(\alpha) = y_0. \end{cases} \tag{1.2}$$

If $p, q \in C[\alpha, b]$ then (1.2) has a unique solution, which can be computed as:

$$y(t) = e^{\int_\alpha^b p(s)\,ds} \left[y_0 + \int_\alpha^b q(s)\, e^{-\int_\alpha^s p(z)\,dz}\, ds \right] \tag{1.3}$$

for $\alpha \le t \le b$.

therefore

$$y(t) = e^{\int_\alpha^b p(s)ds} y_0 + \int_\alpha^t q(s) e^{\int_s^t p(z)dz} ds \qquad \alpha \le t \le b.$$

Prove of (1.3)

Let $p = 0$, then the problem may be computed with a simple integration.

Let $p \neq 0$, then we multiply the differential equation with a integration parameter, so as to able to perform the integration, We have

$$y'(s) - p(s)\, y(s) = q(s) \qquad \alpha \le s \le b.$$

$$\Leftrightarrow \quad e^{-\int_\alpha^s p(z)dz} \left(y'(s) - p(s) y(s) \right) = e^{-\int_\alpha^s p(z)dz} q(s).$$

$$\Leftrightarrow \quad \left(e^{-\int_\alpha^s p(z)dz} y(s) \right)' = e^{-\int_\alpha^s p(z)dz} q(s).$$

Integrating the above relation from a to t, we get the desired result.

Question : We have proved the existense an uniqueness of (1.1) when f is a polynomial of degree at most one with respect to y. What happens for a general f ?

▶ It is not possible always to find a closed formula for the solution.

For example :

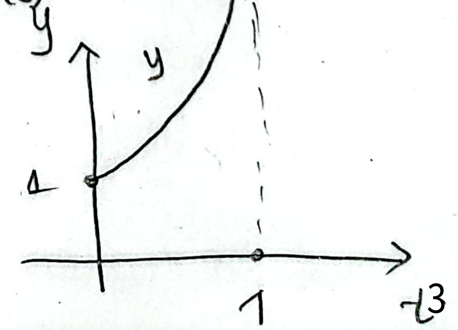
$$\begin{cases} y'(t) = y^2(t) & , 0 \le t \le 2. \\ y(0) = 1. \end{cases}$$

Remark : Since the derivative of y is $y^2 \geq 0$ the the solution is increasing. , and $y(t) \geq 0$, $0 \le t \le 2$.

For the solution

$$\frac{y'(t)}{y^2(t)} = 1 \iff -\frac{d}{dt}\frac{1}{y(t)} = 1.$$

→ integrate from 0 to t $\iff -\frac{1}{y(t)} + \frac{1}{y(0)} = t \iff y(t) = \frac{1}{1-t}$



Thus, the only solution on $0 \le t < 1$ is

$$y(t) = \frac{1}{1-t}$$

But

$$y(t) \to \infty \quad \text{as} \quad t \to 1^-$$

Thus, we don't have solution on $[0,2]$.

Another Example

$$\begin{cases} y'(t) = \sqrt{|y(t)|} & , \ 0 \le t \le 1 \\ y(0) = 0. \end{cases}$$

This IVP has multiple solutions.

$$y(0) = 0 \ , \ 0 \le t \le 1 \quad \text{and} \quad y(t) = \begin{cases} 0 & , \ 0 \le t \le \frac{1}{2}. \\ \frac{(t-\frac{1}{2})^2}{4} & , \ \frac{1}{2} < t \le 1 \end{cases}$$

Theorem 1.1 (Existence and Uniqueness for ODEs).

Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ a continuous function that satisfies the condition of Lipschitz with respect to y and uniformly to t, i.e.,

(1.4) $\exists L \geq 0 \ \forall t \in [a,b] \ \forall y_1, y_2 \in \mathbb{R}$

s.t $$|f(t,y_1) - f(t,y_2)| \le L \, |y_1 - y_2|.$$

Then, for all $y_0 \in \mathbb{R}$, the IVP (1.1) has a unique solution.

Proof

Define the integral operator. $T: C[\alpha,b] \to C[\alpha,b]$

$$Tx(t) := (Tx)(t) := y_0 + \int_\alpha^t f(s,x(s))\,ds \qquad \alpha \le t \le b.$$

Let y a solution of (1.1). Integrate on $[\alpha,t]$ both sides of differential equation of (1.1), we get that

(1.5) $$y = Ty$$

i.e., y is also a solution of the integral equation.

Reverse: If $y \in C[\alpha,b]$ a solution of an integral equation (1.5), then $y(\alpha) = y_0$.

Since f is continuous on $[\alpha,b] \times \mathbb{R}$, then y is also continiusly differentiable on $[\alpha,b]$. Differenting both sides of (1.5), we get that

$$y'(t) = f(t,y(t)) \qquad \text{on } t \in [\alpha,b].$$

$\Rightarrow$ y is the solution of (1.1)

Therefore, to prove that (1.1) has a unique solution, it is sufficient to prove that (1.5) has a unique solution, which is also equivalent to show that T has a unique fixed point

Recall the norm on $C[\alpha,b]$, the $\|\cdot\|$

$$\|x\| := \max_{\alpha \le t \le b} \left(|x(t)| e^{-2Lt} \right)$$

$\rightarrow$ The $\|\cdot\|$ is equivalent do $\|\cdot\|_\infty$ (ex.).

Thus, the space $(C[\alpha,b], \|\cdot\|_\infty)$ is complete and $(C[\alpha,b], \|\cdot\|)$ Banach.

We need to show that T is contraction on $(C[\alpha,b], \|\cdot\|)$, then from Banach fixed point theorem, T have a unique fixed point.

Let $x, z \in C[\alpha,b]$, $t \in [\alpha,b]$

$$|Tx(t) - Tz(t)| = \left| \int_\alpha^t \left(f(s,x(s)) - f(s,z(s)) \right) ds \right|$$

$$\le \int_\alpha^t |f(s,x(s)) - f(z,z(s))| \, ds.$$

$$\le L \int_\alpha^t |x(s) - z(s)| \, ds.$$

Thus

$$|Tx(t) - Tz(t)| \le L \int_\alpha^t |x(s) - z(s)| e^{-2Ls} e^{2Ls} \, ds.$$

$$\le L \|x - z\| \int_\alpha^t e^{2Ls} \, ds \le \frac{1}{2} e^{2Lt} \|x - z\|$$

This means that

$$\| Tx - Tz \| \leq \frac{1}{2} \| x - z \| \quad \forall\, x, z \in C[a,b]$$

We can also prove the result for $\|\cdot\|_\infty$, but we need to make sure that $L(b-a) < 1$.

Remark

The Lipschitz condition is very restrictive. For example,

$$f(t,y) = y^2.$$

This function does not satisfies Lipschitz conditions.

If f is differentiable with respect to its second variable then

$$\begin{cases} \exists\, M \in \mathbb{R} \;\; \forall\, t \in [a,b] \;\; \forall\, y \in \mathbb{R} \quad |f_y(t,y)| \leq M. \\ \text{Then, } f \text{ satisfies Lipschitz condition with } L := M. \end{cases}$$

The condition (1.4) is called global Lipschitz condition, since it holds for all y_1, y_2. If it is holds, we have unique solution on the whole interval $[a,b]$. We can replace the global Lipschitz condition for a weaker, a local, where we can prove existence and uniqueness on $[a,b']$, with b' sufficient

Theorem 1.2 (Local existence and Uniqueness of ODEs)

Let $c>0$ and $f \in C([\alpha,b] \times [y_0 - c, y_0 + c])$. If f satisfies on $[\alpha,b] \times [y_0-c, y_0+c]$ the Lipschitz condition with respect to y and uniformly on t, i.e.,

(1.6) $\exists\, L \geq 0 \ \forall t \in [\alpha,b] \ \forall y_1, y_2 \in [y_0-c, y_0+c]$.

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|.$$

then (1.1) has a unique solution, at least on $[\alpha, b']$, where

$$\begin{cases} b' := \min\left(b, \ \alpha + \frac{c}{A}\right) \\ A = \max\limits_{\substack{\alpha \leq t \leq b \\ y_0 - c \leq y \leq y_0 + c}} |f(t,y)| \end{cases}$$

1.2. Stability

Let f a continuous function.

For initial values $y_0, z_0 \in \mathbb{R}$, we recall the IVPs.

(1.7). $\begin{cases} y' = f(t,y), & \alpha \leq t \leq b. \\ y(\alpha) = y_0 \end{cases}$

(1.8) $\begin{cases} z' = f(t,z), & \alpha \leq t \leq b. \\ z(\alpha) = z_0. \end{cases}$

We will study the behavior of $\|y-z\|$ when $|y_0-z_0|$ is small.

1^{st} case (f satisfies the global Lipschitz condition).

From Theorem 1.1, both (1.7) and (1.8) have unique solutions $y, z \in C^1[a,b]$.

Define $\quad \varepsilon(t) := y(t) - z(t), \quad a \le t \le b.$

$$\Rightarrow \quad \varepsilon'(t) = y'(t) - z'(t) = f(t,y(t)) - f(t,z(t)).$$

Goal: To estimate $\|y-z\|$ or $\|\varepsilon(t)\|$.

Multiply with $\varepsilon(t)$ both sides we get:

$$\varepsilon(t)\,\varepsilon'(t) = (f(t,y(t)) - f(t,z(t)))\,\varepsilon(t).$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(\varepsilon(t))^2 = (f(t,y(t)) - f(t,z(t)))\,\varepsilon(t)$$

$$\le |f(t,y(t)) - f(t,z(t))|\,|\varepsilon(t)|$$

$$\le L|\varepsilon(t)|^2.$$

Let $g(t) := \varepsilon^2(t)$, then

$$g'(t) - 2Lg(t) \le 0, \quad a \le t \le b$$

Multiply with e^{-2Lt} then

$$e^{-2Lt} g'(t) - 2L e^{-2Lt} g(t) = \frac{d}{dt}\left(e^{-2Lt} g(t)\right) \le 0$$

$$\alpha \le t \le b$$

Thus $e^{-2Lt} g(t)$ is a decreasing on $[\alpha, b]$ and thus.

$$e^{-2Lt} g(t) \le e^{-2L\alpha} g(\alpha) \qquad \alpha \le t \le b.$$

$$\Rightarrow \quad g(t) \le e^{+2L(t-\alpha)} g(\alpha).$$

$$\le e^{2L(b-\alpha)} g(\alpha)$$

$$\Rightarrow \quad \varepsilon^2(t) \le e^{2L(b-\alpha)} \varepsilon^2(\alpha).$$

$$\Rightarrow \quad |\varepsilon(t)| \le e^{L(b-\alpha)} |\varepsilon(\alpha)|. \qquad \alpha \le t \le b$$

Thus

$$|y(t) - z(t)| \le e^{L(b-\alpha)} |y_0 - z_0| \quad , \quad \alpha \le t \le b$$

This hold also for maximum,

$$\max_{\alpha \le t \le b.} |y(t) - z(t)| \le e^{L(b-\alpha)} |y_0 - z_0|$$

$$\Rightarrow \quad \boxed{\|y - z\|_\infty \le e^{L(b-\alpha)} \|z_0 - y_0\|_\infty}$$

2st case (f satisfies the one-side Lipschitz condition).

One-side Lipschitz condition

(1.7) $\forall t \in [\alpha, b] \quad \forall y_1, y_2 \in \mathbb{R} \quad (f(t, y_1) - f(t, y_2))(y_1 - y_2) \leq 0.$

Again

$$\varepsilon'(t)\varepsilon(t) = (f(t, y(t)) - f(t, z(t)))\varepsilon(t) \leq 0$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\varepsilon^2(t) \leq 0.$$

Thus $\varepsilon^2(t)$ is a decreasing function w.r.t t. This also hold for $|\varepsilon(t)|$, thus

$$\max_{a \leq t \leq b} |y(t) - z(t)| \leq |y_0 - z_0|.$$

From the above inequality we have the existence and uniqueness of (1.1) in case where f satisfies (1.7).

1.3 Systems of ODEs.

It is very important to generalize the systems of ODEs of first order. (for)

Let $m \in \mathbb{N}$, $f : [\alpha, b] \times \mathbb{R}^m \to \mathbb{R}^m$, $y_0 \in \mathbb{R}^m$

We seek $y : [\alpha, b] \to \mathbb{R}^m$ s.t.

(1.8) $$\begin{cases} y'(t) = f(t, y(t)), & \alpha \leq t \leq b \\ y(\alpha) = y_0. \end{cases}$$

All the results of Sections (1.1), (1.2) holds also now, but we need to replace the absolute value, with a $\|\cdot\|$ norm of $\mathbb{R}^m$.

Theorem 1.3 (Existence and Uniqueness for 1.8)

Let that $f : [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$ a continuous function, that satisfies the Lipschitz condition with respect to y, uniformly on t, in norm $\|\cdot\|$ of $\mathbb{R}^m$, i.e.,

(1.9) $\exists\, L \in \mathbb{R} \;\; \forall t \in [a,b], \; \forall y_1, y_2 \in \mathbb{R}^m$

$$\|f(t,y_1) - f(t,y_2)\| \leq L \|y_1 - y_2\|.$$

2. Numerical Methods.

One can distinguish between analytic approximation methods and discrete-variable methods. In the former, one tries to find approximations

$$y_a(t) \approx y(t).$$

$\hookrightarrow$ exact sol.

which are valid for all $t \in [a,b]$

In discrete-variable methods, one attempts to find approximations

$$Y^n \in \mathbb{R}^m \quad \text{of} \quad y(t^n)$$

where $\quad Y^n \approx y(t^n) \quad , \; n = 0, \ldots\ldots, M$